Скрытая масса
как следствие теоремы вириала Пуанкаре

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$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i \quad (i=1,2,\ldots,n), \quad \ldots \quad (1)$$

$$m_i\left(\vec{r}_i \cdot \frac{d^2\vec{r}_i}{dt^2}\right) = (\vec{r}_i \cdot \vec{F}_i), \quad \ldots \quad (2)$$

$$\sum_{i=1}^{n} m_i\left(\vec{r}_i \cdot \frac{d^2\vec{r}_i}{dt^2}\right) = \sum_{i=1}^{n}(\vec{r}_i \vec{F}_i), \quad \ldots \quad (3)$$

$$\left(\vec{r}_i \cdot \frac{d^2\vec{r}_i}{dt^2}\right) = \frac{d}{dt}\left(\vec{r}_i \frac{d\vec{r}_i}{dt}\right) - \left(\frac{d\vec{r}_i}{dt}\right)^2, \quad \ldots \quad (4)$$

$$\left(\vec{r}_i \frac{d\vec{r}_i}{dt}\right) = \frac{1}{2}\frac{d}{dt}\left(\vec{r}_i^{\,2}\right), \quad \ldots \quad (5)$$

$$\left(\vec{r}_i \frac{d^2\vec{r}_i}{dt^2}\right) = \frac{1}{2}\frac{d^2\vec{r}_i}{dt^2} - \left(\frac{d\vec{r}_i}{dt}\right)^2, \quad \ldots \quad (6)$$

$$\frac{1}{2}\sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i}{dt^2} - \sum_{i=1}^{n} m_i\left(\frac{d\vec{r}_i}{dt}\right)^2 = (\vec{r}_i \vec{F}_i) \quad \ldots \quad (7)$$

$$J \equiv \sum_{i=1}^{n} m_i \vec{r}_i^{\,2}, \quad \ldots (8)$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\frac{d\vec{r}_i}{dt}\right)^2 \quad \ldots (9)$$

$$V = \sum_{i=1}^{n}(\vec{r}_i \cdot \vec{F}_i) \quad \ldots (10)$$

$$\frac{1}{2}\frac{d^2 J}{dt^2} = V + 2T \quad \ldots \quad (11)$$

1). $\dfrac{d^2 J}{dt^2} = 0 \ \ldots (12)$ $\qquad \overline{T} = -\dfrac{1}{2}\overline{V} \ \ldots (13)$

(теорема вириала, Клаузиус, 1870)

$$\vec{F}_i = -\frac{\partial \Omega}{\partial \vec{r}_i}, \quad \Omega = -G\sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{m_i m_j}{r_{ij}}, \quad r_{ij} = |\vec{r}_j - \vec{r}_i| \quad \ldots (14)$$

$$\Omega(\ell\vec{r}_i, \ell\vec{r}_j) = \ell^{-1}\Omega(\vec{r}_i, \vec{r}_j), \quad \sum_{i=1}^{n}\left(\vec{r}_i \frac{\partial \Omega}{\partial \vec{r}_i}\right) = -\Omega \quad \ldots (15)$$

$$V = -\sum_{i=1}^{n}\left(\vec{r}_i \frac{\partial \Omega}{\partial \vec{r}_i}\right) = +\Omega \quad \ldots (16) \qquad 2\overline{T} = -\overline{\Omega}$$ (Пуанкаре, 1913) ... (17)

2). $\dfrac{d^2 J}{dt^2} \neq 0 \ \ldots (18)$, $\qquad \dfrac{1}{2}\dfrac{d^2 J}{dt^2} = 2T + \Omega \quad \ldots (19)$

(Эддингтон, 1916)

$$(20)\ldots\quad R_{ij}-\frac{1}{2}g_{ij}R-\Lambda g_{ij}=\varkappa T_{ij},\quad \varkappa>0,\ \Lambda>0,\ (i,j=0,1,2,3)$$

$$(21)\ldots\quad R_{ij}=\frac{\partial\Gamma^{l}_{ij}}{\partial x^{l}}-\frac{\partial\Gamma^{l}_{lj}}{\partial x^{i}}+\Gamma^{l}_{ij}\Gamma^{m}_{lm}-\Gamma^{m}_{il}\Gamma^{l}_{jm},\quad R=g^{ij}R_{ij}$$

$$(22)\ldots\quad \Gamma^{l}_{ij}=\Gamma_{m,ij}g^{lm},\quad \Gamma_{m,ij}=\frac{1}{2}\left(\frac{\partial g_{mi}}{\partial x^{j}}+\frac{\partial g_{mj}}{\partial x^{i}}-\frac{\partial g_{ij}}{\partial x^{m}}\right)$$

$$(23)\ldots\quad g_{ij}=g_{ji},\quad g^{il}g_{lk}=\delta^{i}_{k},\quad \delta^{i}_{k}=\begin{cases}1 & (i=k)\\ 0 & (i\neq k)\end{cases}$$

$$(24)\quad ds^{2}=g_{ik}dx^{i}dx^{k},\quad x^{0}\equiv ct,\ x^{1}\equiv r,\ x^{2}\equiv\theta,\ x^{3}\equiv\varphi$$

(А.Эйнштейн, 1916)

$$(25)\ldots\quad \begin{cases} g_{00}=1-\dfrac{2GM}{c^{2}r}-\dfrac{1}{3}\Lambda r^{2}, & g_{ik}=0\ (i\neq k)\\[2mm] g_{11}=-\left(1-\dfrac{2GM}{c^{2}r}-\dfrac{1}{3}\Lambda r^{2}\right)^{-1}, & g_{22}=-r^{2},\ g_{33}=-r^{2}\sin^{2}\theta \end{cases}$$

$$(26)\ldots\quad g_{00}=1-\frac{2\psi}{c^{2}},\quad \psi=\frac{GM}{r}+\frac{1}{6}\Lambda c^{2}r^{2}$$

$$(27)\ldots\quad W=-\Omega+B$$

$$(28)\ldots\quad B=\frac{1}{6}\Lambda c^{2}\sum_{i=1}^{n}\sum_{j=1}^{n}m_{i}r_{ij}^{2}$$

$$(29)\ldots\quad \bar{\Omega}=-G\,C_{n}^{2}\frac{m^{2}}{R}\simeq-G\frac{(nm)^{2}}{2R}=-\frac{GM^{2}}{2R}$$

$$(30)\ldots\quad M=\sum_{i=1}^{n}m_{i}=n\cdot m$$

$$(31)\ldots\quad \bar{B}=\frac{1}{6}\Lambda c^{2}MR^{2}$$

$$(32)\ldots\quad \langle v^{2}\rangle=\frac{1}{M}\sum_{i=1}^{n}m_{i}\left(\frac{d\vec{r}_{i}}{dt}\right)^{2}$$

$$(33)\ldots\quad \frac{1}{2}\frac{d^{2}J}{dt^{2}}=(\bar{\Omega}+2\bar{B})+2\bar{T}$$

$$(34)\ldots\quad \frac{d^{2}J}{dt^{2}}=0,\quad 2\bar{T}=-\bar{\Omega}-2\bar{B}$$

$$(35)\ldots\quad M\langle v^{2}\rangle=+\frac{GM^{2}}{2R}-\frac{1}{6}\Lambda c^{2}MR^{2},$$

$$(36)\quad \langle v^{2}\rangle=\frac{GM}{2R}-\frac{1}{6}\Lambda c^{2}R^{2},$$

$$(37)\quad M=\frac{\Lambda c^{2}}{3G}R^{3}+\frac{2}{G}R\langle v^{2}\rangle,$$

$$(38)\quad \rho_{v}=\frac{\Lambda c^{2}}{8\pi G},\quad M=\frac{2R\langle v^{2}\rangle}{G}+\frac{8}{3}\pi\rho_{v}R^{3}$$

$$(39)\quad M=M_{p}+\Delta M,\quad M_{p}=\frac{2R\langle v^{2}\rangle}{G},\quad \Delta M=\frac{8}{3}\pi\rho_{v}R^{3}$$

$$(40)\quad \rho_{G}=-2\rho_{v},\quad \Delta M=-\frac{4}{3}\pi R^{3}\rho_{G}\equiv M_{v}<0.$$